Cellular automaton rules conserving the number of active sites

1998 J. Phys. A: Math. Gen. 31 6007

(http://iopscience.iop.org/0305-4470/31/28/014)

J. Phys. A: Math. Gen. **31** (1998) 6007–6018. Printed in the UK PII: S0305-4470(98)90420-6

Cellular automaton rules conserving the number of active sites

Nino Boccara† and Henryk Fukś‡
University of Illinois at Chicago, Department of Physics, Chicago, IL 60607-7059, USA

Received 5 January 1998

Abstract. This paper shows how to determine all of the unidimensional two-state cellular automaton rules of a given number of inputs which conserve the number of active sites. These rules have to satisfy a necessary and sufficient condition. If the active sites are viewed as cells occupied by identical particles, these cellular automaton rules represent evolution operators of systems of identical interacting particles whose total number is conserved. Some of these rules, which allow motion in both directions, mimic ensembles of one-dimensional pseudorandom walkers. Numerical evidence indicates that the corresponding stochastic processes might be non-Gaussian.

1. Introduction

Systems which consist of a large number of simple identical elements evolving in time according to simple rules often exhibit a complex behaviour as a result of the cooperative effect of their components. Cellular automata (CA) are models of such systems. They may be defined as follows. Let $s : \mathbb{Z} \times \mathbb{N} \mapsto \{0, 1\}$ be a function that satisfies the equation

$$s(i, t + 1) = f(s(i - r_l, t), s(i - r_l + 1, t), \ldots, s(i + r_r, t)) \tag{1}$$

for all $i \in \mathbb{Z}$ and all $t \in \mathbb{N}$, where $\mathbb{Z}$ is the set of all integers and $\mathbb{N}$ is the set of non-negative integers. Such a discrete dynamical system is a two-state one-dimensional CA. The mapping $f : \{0, 1\}^{r_l+r_r+1} \to \{0, 1\}$ is the rule, and the positive integers r_l and r_r are, respectively, the left and right radius of the rule. f will also be called an n-input rule where $n = r_l + r_r + 1$. The function $S_t : i \mapsto s(i, t)$ is the state of the CA at time t. $\mathcal{S} = \{0, 1\}^{\mathbb{Z}}$ is the state space. An element of the state space is also called a configuration. Since the state S_{t+1} at time $t + 1$ is entirely determined by the state S_t at time t and the rule f, there exists a unique mapping $F_f : \mathcal{S} \to \mathcal{S}$ such that $S_{t+1} = F_f(S_t)$. F_f, which is the evolution operator, is also referred to as the global CA rule.

CA have been widely used to model complex systems in which the local character of the rule plays an essential role (Wolfram 1983, Farmer *et al* 1984, Manneville *et al* 1989, Gutowitz 1990, Boccara *et al* 1993). In the past few years, CA have been successfully used to model highway traffic. One of the simplest models is defined on a one-dimensional lattice of L sites with periodic boundary conditions. Each site is either occupied by a vehicle, or empty. The velocity of each vehicle is an integer between 0 and $v_{\max}$. If $x(i, t)$ denotes the

† E-mail address: boccara@uic.edu
‡ E-mail address: fuks@sunphy1.phy.uic.edu

0305-4470/98/286007+12$19.50

position of car i at time t, the position of the next car ahead at the same time is $x(i+1,t)$. With this notation, the system evolves according to a synchronous rule given by

$$x(i,t+1) = x(i,t) + v(i,t+1) \tag{2}$$

where

$$v(i,t+1) = \min(x(i+1,t) - x(i,t) - 1, x(i,t) - x(i,t-1) + a, v_{\max}) \tag{3}$$

is the velocity of car i at time $t+1$. $x(i+1,t) - x(i,t) - 1$ is the gap (number of empty sites) between cars i and $i+1$ at time t, $x(i,t) - x(i,t-1)$ is the velocity $v(i,t)$ of car i at time t, and a is the acceleration. $a = 1$ corresponds to the deterministic model of Nagel and Schreckenberg (1992) while the case $a = v_{\max}$ was considered by Fukui and Ishibashi (1995). In this case, the evolution rule can be written

$$x(i,t+1) = x(i,t) + \min(x(i+1,t) - x(i,t) - 1, v_{\max}). \tag{4}$$

This is a CA rule with, at least, its left radius equal to $v_{\max}$ and its right one equal to $v_{\max} - 1$. The case $a < v_{\max}$ is a second-order rule, that is, the state at time $t+1$ depends upon the states at times t and $t-1$. For $v_{\max} = 1$, these two rules coincide with the elementary CA rule 184 (rule code numbers as in Wolfram (1994)).

Since, for these highway traffic models on a ring (we shall always consider cyclic boundary conditions), the number of cars is conserved, it might be interesting to address the more general question: is it possible to determine all one-dimensional two-state CA rules which conserve the number of active sites? We cannot expect that all these rules will mimic realistic highway traffic. It is preferable to view them as describing the evolution of systems which consist of a fixed number of interacting particles.

2. General considerations

If the sites are either all inactive or all active, they should remain so during the evolution. Therefore, for any number of inputs n, the local rule should satisfy the conditions

$$f(\underbrace{0,0,0,\ldots,0}_{n}) = 0 \tag{5}$$

$$f(\underbrace{1,1,1,\ldots,1}_{n}) = 1. \tag{6}$$

If the rule (1) changes the site value $s(i,t)$, we may say that it either 'created' a particle, if $s(i,t+1) = 0$ and $s(i,t+1) = 1$, or 'annihilated' a particle in the opposite case. Since, the argument $s(i,t)$ of function f takes the values 0 and 1 an equal number of times, conservation of a particle's number implies that the number of creations and annihilations should be equal. In other words, the number of preimages of 0 and 1 by f should be the same.

Consider rules f_1 and f_2, whose radii are, respectively, r_{l1}, and r_{r1}, and r_{l2} and r_{r2}. The rule $f_1 \circ f_2$ which consists, at each timestep, of the successive application of f_1 and f_2, conserves the number of particles if f_1 and f_2 do. Its radii are $r_l = r_{l1} + r_{l2}$ and $r_r = r_{r1} + r_{r2}$. For instance, the four-input rule whose binary code number is 1011100010111000 ($r_l = 1$, $r_l = 2$) conserves the number of particles since it is the composition of the left shift (binary code number 1010, $r_l = 0$, $r_r = 1$) and rule 184 (binary code number 10111000, $r_l = r_r = 1$) which both conserve the number of particles.

If, as for highway traffic, we wish to follow particles motion, it might be useful to define a representation of rule f which exhibits this motion. Such a 'motion representation' may be defined as follows. List all the neighbourhoods of a given particle represented by **1**. Then, for each neighbourhood, indicate the displacement of this particle by an integer v, where v is positive if the particle moves to the right and negative if it moves to the left. For instance, the motion representation of rule 184 would be

$$\mathbf{1}0\ 1,\quad \mathbf{1}1\ 0. \tag{7}$$

Since, for this particular rule, the particle can only move to the right, we only need to indicate the relevant neighbourhood of the particle. This representation can be made more visual if we draw an arrow joining the initial and final positions of the particle, i.e. for rule 184

$$\overset{\curvearrowright}{10},\quad \overset{\circlearrowright}{1}1. \tag{8}$$

Note that, in this case, it is not necessary to specify the moving particle by a bold digit.

This last notation is very compact. For instance, the four-input rule which results from the composition of rule 184 and the left shift, is represented by

$$\overset{\circlearrowright}{1}0,\quad \overset{\curvearrowleft}{\bullet 1}1$$

where $\bullet$ represents either 0 or 1. The motion representation has another advantage. When we are interested by the motion of the particles, the knowledge of the rule table, which gives the images of the various n-inputs, is not sufficient. We have to specify the values of the right and left radii since modifying r_l and r_r at constant n is equivalent to adding a constant velocity to all the particles.

Rules obtained by reflection or conjugation of a rule conserving the number of active sites have the same property. Reflection exchanges the values of r_l and r_r and changes the sign of the velocity. Conjugation exchanges the roles of 0's and 1's, that is, if a rule describes a specific motion of particles (represented by 1's) then its conjugate describes the same rule, but for the motion of holes (represented by 0's). If R and C denote, respectively, these two operators, two n-input rules f_1 and f_2 are said to be equivalent if there exists an element g of the four-group generated by R and C which transforms f_1 into f_2.

3. Rules determination

One method to determine all of the n-input rules f conserving the number of active sites is to find a system of equations whose solutions are all the functions

$$f : \{0, 1\}^n \mapsto \{0, 1\} \tag{9}$$

which, for all $L \geqslant n$, satisfy the conditions

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) + f(x_2, x_3, \ldots, x_{n+1}) + \cdots + f(x_L, x_1, \ldots, x_{n-1}) \\ = x_1 + x_2 + \cdots + x_L \end{aligned} \tag{10}$$

for all L-ring configurations (cyclic permutations). Such a system shall be called an L-system of equations. Conditions (10) are clearly necessary, but does there exist as a minimum value $L_{\min}$ of L such that they are also sufficient?

We shall prove that $L_{\min}$ *exists, and is equal to* $2n - 2$. That is, *the necessary and sufficient condition for a rule f to conserve the number of active sites is to satisfy relations (10) for $L = 2n - 2$.*

Before giving a formal proof of this result, we shall present a simple, but not rigorous, argument. Given the states $s(1, t), s(2, t), \ldots, s(n, t)$ of sites $1, 2, \ldots, n$ at time t, the state $s(r_{r_l+1}, t + 1)$ of site $r_l + 1$ at time $t + 1$ is determined ($n = r_l + r_r + 1$). To determine the states of sites 1 and n at time $t+1$, we also need to know the states at time t of the r_l sites on the left of site 1 and the r_r sites on the right of site n. To obtain the minimum number of sufficient conditions satisfied by (9), we shall require that the minimum number of sites we have to add to the original n sites should be such that their state values at time $t+1$ should depend on, at least, one of the site values $s(1,t), s(2,t), \ldots, s(n,t)$. This condition implies that we should consider an $L_{\min}$-ring in which the sites $1-r_l$ and $n+r_r$ coincide. Therefore, $L_{\min} = r_l + n + r_r - 1$, that is, $L_{\min} = 2n-2$.

To prove the above result in a more rigorous way, we shall show that, if $L > 2n-2$, any equation of an L-system is a linear combination of three equations belonging, respectively, to $(L-1)$-, $(2n-3)$-, and $(2n-2)$-systems. More precisely, for all L-ring configurations $\{x_1, x_2, \ldots, x_L\}$, equation (10) can be written

$$\begin{aligned}
&(f(x_1, x_2, \ldots, x_n) + f(x_2, x_3, \ldots, x_{n+1}) + \cdots + f(x_{L-1}, x_1, \ldots, x_{n-1}))\\
&\qquad -(f(x_1, x_2, \ldots, x_{n-2}, x_{L-n+1}, x_{L-n+2})\\
&\qquad +f(x_2, x_3, \ldots, x_{L-n+3}) + \cdots + f(x_{n-2}, x_{L-n+1}, \ldots, x_{L-1})\\
&\qquad +f(x_{L-n+1}, x_{L-n+2}, \ldots, x_{L-1}, x_1) + \cdots + f(x_{L-1}, x_1, \ldots, x_{n-2}, x_{L-n+1}))\\
&\qquad +(f(x_1, x_2, \ldots, x_{n-2}, x_{L-n+1}, x_{L-n+2})\\
&\qquad +f(x_2, x_3, \ldots, x_{L-n+3}) + \cdots + f(x_{n-2}, x_{L-n+1}, \ldots, x_{L-1})\\
&\qquad +f(x_{L-n+1}, x_{L-n+2}, \ldots, x_{L-1}, x_L) + \cdots + f(x_{L-1}, x_L, x_1, \ldots, x_{n-2})\\
&\qquad +f(x_L, x_1, \ldots, x_{n-2}, x_{L-n+1}))\\
&\quad = (x_1 + x_2 + \cdots + x_{L-1}) - (x_1 + \cdots + x_{n-2} + x_{L-n+1} + \cdots + x_{L-1})\\
&\qquad +(x_1 + \cdots + x_{n-2} + x_{L-n+1} + \cdots + x_L).
\end{aligned} \tag{11}$$

To verify this result, we have to assume that $x_{L-1} = x_L$, which is always the case for any cycle, except when L is even, for the cycle $1010\ldots10$. Verifying (11) is then a bit tedious but straightforward. By induction, relation (11) shows that any equation of an L-system is a linear combination of equations belonging to $(2n-3)$- and $(2n-2)$-systems.

The equation corresponding to the cyclic configuration $1010\ldots10$ reads

$$\underbrace{f(1010\ldots10) + f(0101\ldots01) + \cdots + f(0101\ldots01)}_{L} = \frac{L}{2} \tag{12}$$

if n is even, and

$$\underbrace{f(1010\ldots01) + f(0101\ldots10) + \cdots + f(0101\ldots10)}_{L} = \frac{L}{2} \tag{13}$$

if n is odd. That is,

$$f(1010\ldots10) + f(0101\ldots01) = 1 \tag{14}$$

if n is even, and

$$f(1010\ldots01) + f(0101\ldots10) = 1 \tag{15}$$

if n is odd. One of the images by f of the two alternating n-sequences of 0's and 1's is equal to 1, and the other one to 0.

4. Examples

One- and two-input rules conserving the number of active sites are trivial. The identity, represented by $\overset{\circlearrowleft}{1}$, is the only one-input rule, and the left and right shifts, represented respectively by $\bullet\overset{\curvearrowleft}{1}$ and $\overset{\curvearrowright}{1}\bullet$, are the only two-input rules. Note that the rule represented by $\overset{\circlearrowright}{1}\bullet$ or $\bullet\overset{\circlearrowright}{1}$ is the identity viewed as a two-input rule, but in agreement with our convention to only represent the relevant neighbourhood, we shall always represent it as a one-input rule. This is a general feature. When we solve the system of equations (10) for $n = 3$ and $L_{\text{min}} = 4$, we shall re-obtain the identity, and the left and right shifts as three-input rules.

4.1. Three-input rules

The only non-trivial three-input rules conserving the number of active sites are rules 184 and 226, represented respectively by

$$\overset{\curvearrowright}{1}0,\ \overset{\circlearrowright}{1}1 \qquad \text{and} \qquad 0\overset{\curvearrowleft}{1},\ 1\overset{\circlearrowright}{1}. \tag{16}$$

Rule 226, which can be obtained either by reflection or conjugation of rule 184, models exactly the same deterministic highway traffic rule. The only difference, clearly shown by the motion representation, is that cars move to the right instead of moving to the left.

4.2. Four-input rules

The system of equations (10) for $n = 4$ and $L_{\text{min}} = 6$ has 22 solutions. Among these, we re-obtain the identity, the left and right shifts, rules 184 and 226 and some simple combinations of these rules viewed as four-input rules. The new rules are as follows.

- Rules 43 944, 65 026, 59 946, 49 024. The motion representation of rule 43 944 ($r_l = 2, r_r = 1$) is

$$\overset{\curvearrowright}{1}00,\ \overset{\curvearrowright}{1}01,\ \overset{\circlearrowright}{1}1.$$

This rule coincides with the highway traffic rule (4) for $v_{\text{max}} = 2$, and cars moving to the right. Rule 65 026, obtained by reflection of 43 944, describes the same highway traffic rule but for cars moving in the opposite direction.

The motion representation of rule 59 946, which is the conjugate of rule 43 944, is

$$01\overset{\curvearrowleft}{1},\ 10\overset{\curvearrowleft}{1},\ 11\overset{\circlearrowright}{1}.$$

It describes a highway traffic rule in which drivers, anticipating the motion of the car ahead, may move to an occupied site with $v_{\text{max}} = 1$. More general rules of this type have been studied by Fukś and Boccara (1997). Rule 49 024 is obtained by reflection of rule 59 946.

- Rules 58 336, 52 930, 63 544, 48 268. The motion representation of rule 58 336 ($r_l = 1, r_r = 2$) is

$$\overset{\curvearrowright}{1}00,\ \overset{\circlearrowright}{1}01,\ \overset{\circlearrowright}{1}1.$$

It describes a highway traffic rule of overcautious drivers who move to the right with a velocity equal to 1 if, and only if, they have two empty sites ahead of them. By reflection we obtain rule 52 930 describing the same highway traffic rule but for cars moving in the opposite direction.

The motion representation of rule 63 544, conjugate of rule 58 336, is

$$0\overset{\curvearrowleft}{1}1,\ 0\overset{\circlearrowright}{1}0,\ 1\overset{\circlearrowright}{1}.$$

A particle moves to the left if, and only if, the neighbouring left site is empty, and the neighbouring right site is occupied. If the neighbouring left site is occupied the particle does not move. As a highway traffic rule, it describes drivers who do not like to be followed, and move to an empty site only when there is a car just behind them. By reflection we obtain rule 48 268.

• Rules 56 528, 57 580, 62 660, 51 448. The motion representation of rule 56 528 ($r_l = 1, r_r = 2$) is

$$\overset{\circlearrowright}{1}00,\quad \overset{\curvearrowright}{1}01,\quad \overset{\circlearrowright}{1}1.$$

A particle moves to the right if, and only if, its first right site is empty and its second right site is occupied. As a highway traffic rule it describes drivers who move to an empty site if, as a result, they can be just behind another car. Rule 57 580 is obtained by reflection.

The motion representation of rule 62 660, conjugate of rule 56 528, is

$$0\overset{\curvearrowleft}{1}1,\quad 0\overset{\circlearrowright}{1}1.$$

The particle moves to an empty site on its left if, and only if, there is an empty site on its right. Rule 51 448 is obtained by reflection.

• Rules 60 200, 48 770. These rules are self-conjugate. The motion representation of rule 60 200 ($r_l = 1, r_r = 2$) is

$$\overset{\curvearrowright}{1}00,\quad \overset{\circlearrowright}{1}01,\quad 0\overset{\curvearrowleft}{1}1,\quad 1\overset{\circlearrowright}{1}1.$$

A particle moves to the right if, and only if, it has two neighbouring empty sites on that side. If only the first neighbouring site is empty, it does not move to avoid occupying a site close to another particle. If its first right neighbouring site is occupied, then the particle moves to the left when that site is empty. The effective interaction between these particles is repulsive. Rule 48 770, obtained by reflection, describes a similar evolution rule.

These last two rules have interesting properties. Starting from a random initial configuration, after a maximum number of timesteps equal to $N/2$, where N is the number of sites, the system evolves on its limit set. This limit set has a rather simple structure. If the density of particles $\rho = \frac{1}{2}$, it consists of three types of periodic sequences, namely:

$$\begin{aligned} &\ldots 101010101010 \ldots && \text{of period 2} \\ &\ldots 100100100100 \ldots && \text{of period 3} \\ &\ldots 110110110110 \ldots && \text{of period 3.} \end{aligned}$$

The probabilities of the various three-blocks have been determined numerically. We have found

$$\begin{aligned} &P(000) = P(111) = 0 \\ &P(001) = P(110) = P(100) = P(011) = 0.145 \pm 0,001 \\ &P(010) = P(101) = 0.210 \pm 0.001. \end{aligned}$$

Regarded as a formal language (Wolfram 1984, Denning *et al* 1978, Hopcroft and Ullmam 1986), such a limit set is regular. Words in a regular language, on the alphabet $\{0, 1\}$, are generated by walks through a finite directed graph whose arcs are labelled with 0 or 1. Given a finite graph, it is always possible to find an equivalent deterministic finite graph, that is, a graph in which no more than one arc of a given label leaves each vertex. For rules 60 200 and 48 770, the corresponding deterministic graph is represented in figure 1. For $\rho = \frac{1}{2}$, when the CA evolves on its limit set, each particle performs a pseudorandom walk. The CA rules being deterministic, the randomness comes from the randomness of the initial configuration. Numerical simulations show that any particle has a probability $p = 0.29$ to move either to the left or right, and a probability $q = 1 - 2p = 0.42$ not

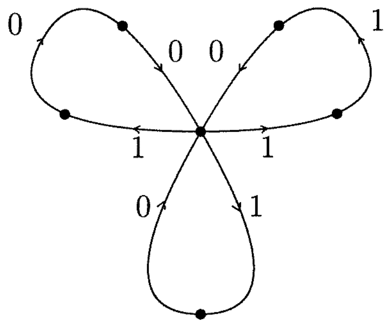

Figure 1. Regular language graph for rules 60 200 and 48 770.

to move. Actually this pseudorandom motion is periodic in time, the period being equal to $N/2$. In the limit set, for a given random initial configuration, all particles perform the same pseudorandom walk, with a phase difference depending on the distance separating them. More precisely, if $X_n(t)$ denotes the position of particle n at time t, for rule 60 200, we have

$$X_n(t) = X_{n+1}(t-1) - 2$$

which implies

$$X_n(t) = X_{n+t}(0) - 2t.$$

This last result shows that the position of a specific particle at time t is determined by the position of another specific particle in the initial configuration.

To characterize the nature of the randomness of the motion of a particle, we have determined the Hurst exponent (Hurst 1951, Hurst *et al* 1965, Feder 1988) of the time series generated by the displacement of a given particle. Given a time series $s(t)$, the Hurst exponent H characterizes the asymptotic behaviour of the standard deviation of $s(t)$ as a function of time. A Brownian motion (symmetric random walk) has a Hurst exponent $H = \frac{1}{2}$. For a particle moving according to the four-input rules 60 200 and 48 770, we have found $H = 0.63 \pm 0.02$. Since the pseudo-random motion is periodic in time with a period equal to half the lattice size, this numerical result is debatable. It could be interesting to perform a detailed study of the correlations, but, even correlated random walks may have a Gaussian behaviour when the number of timesteps goes to infinity (Weiss 1994). However, in this case, there exists a crossover between a non-Gaussian and a Gaussian behaviour. This fact implies that, for a large value of the number of timesteps t, the exponent of the standard deviation of the walk could, numerically, be different from $\frac{1}{2}$.

When $\rho \neq \frac{1}{2}$, the limit set consists of the previous periodic sequences and either sequences of 0's if $\rho < \frac{1}{2}$ or sequences of 1's if $\rho > \frac{1}{2}$. From the motion representation of rule 60 200, it follows that the average velocity $\langle v \rangle$ of the particles is $P(100) - P(011)$. The conjugacy operator changes $\langle v \rangle$ in $-\langle v \rangle$ and ρ in $1-\rho$. Therefore, the so-called 'fundamental diagram' of road traffic theory, that is, the graph of the flow $\rho\langle v \rangle$ as a function of the density ρ, has a centre of symmetry, namely, the point $(\rho, \rho\langle v \rangle) = (\frac{1}{2}, 0)$ (figure 2). Rule 48 770 has identical properties.

4.3. Five-input rules

The number of rules conserving the number of active sites grows very fast with the number of inputs. There exist 428 five-input rules conserving the number of active sites. A few of them are not new either because they actually depend upon a smaller number of inputs

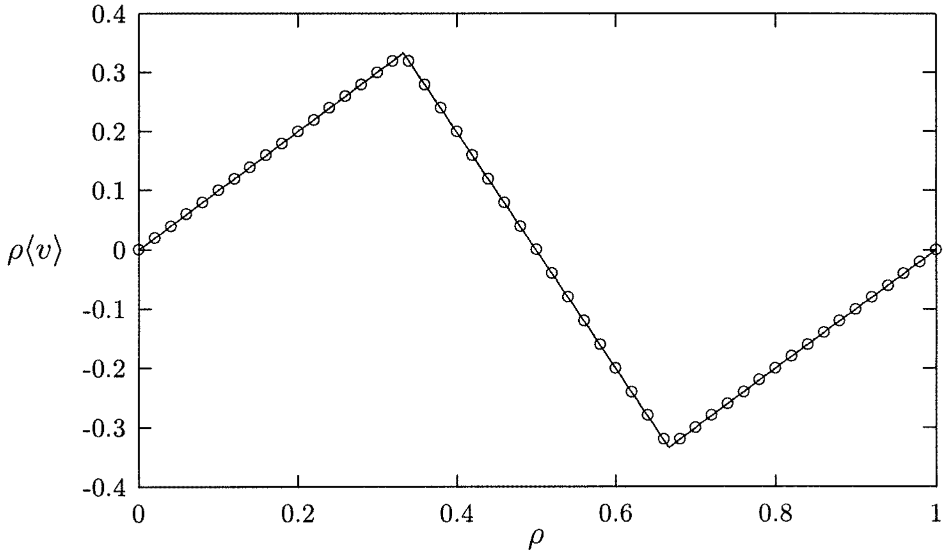

Figure 2. Fundamental diagram for rule 60 200. Small circles represent numerical results. The piecewise linear line has been obtained using local structure approximation (see below).

or because they are a simple composition of the rules already obtained. In this section we shall just describe the self-conjugate rules†.

There exists 20 self-conjugate rules. Some, such as the identity and the shifts (left and right, simple and double), are trivial. We also re-obtain the two self-conjugate four-input rules. Each of them twice depending on which side, left or right, the extra input is added. Finally, we are left with 11 new self-conjugate rules. For each rule we shall always choose the values of r_l and r_r such that the condition $\langle v\rangle = 0$ for $\rho = \frac{1}{2}$ is satisfied. This can always be done.

Few of these rules are still not very interesting. After few timesteps, for all $\rho \in]0, 1[$, five rules emulate the identity, which means that no particles are moving. These rules are: rule 346 456 0268 ($r_l = 1$, $r_r = 3$), whose motion representation is

$$000\overset{\circlearrowright}{1},\quad 01\overset{\circlearrowright}{1}1,\quad 10\overset{\curvearrowleft}{01},\quad 010\overset{\circlearrowright}{1},\quad 01\overset{\curvearrowright}{10},\quad 11\overset{\circlearrowright}{1},\quad 110\overset{\circlearrowright}{1}$$

rule 377 126 4248 ($r_l = 1$, $r_r = 3$), whose motion representation is

$$00\overset{\circlearrowright}{1}1,\quad 00\overset{\curvearrowright}{10},\quad 001\overset{\circlearrowright}{1},\quad 010\overset{\circlearrowright}{1},\quad 11\overset{\curvearrowleft}{01},\quad 11\overset{\circlearrowright}{1},\quad 101\overset{\circlearrowright}{1}$$

and rule 382 473 8360 ($r_l = r_r = 2$), whose motion representation is

$$0\overset{\curvearrowleft}{01}1,\quad 00\overset{\circlearrowright}{1}0,\quad 1\overset{\circlearrowright}{1}01,\quad 01\overset{\circlearrowright}{1}1,\quad 10\overset{\circlearrowright}{1}\,1\overset{\curvearrowright}{10}0,\quad 11\overset{\circlearrowright}{1}1.$$

Rules 424 966 8928 and 415 766 320 obtained by reflection of the first two rules have the same property. Rule 382 473 8360 is invariant under reflection.

Rule 316 765 3058 ($r_l = 3$, $r_r = 1$), whose motion representation is

$$00\overset{\curvearrowleft}{01},\quad 01\overset{\circlearrowright}{1},\quad 10\overset{\circlearrowright}{1},\quad 11\overset{\circlearrowright}{1}1,\quad 100\overset{\circlearrowright}{1},\quad 11\overset{\curvearrowright}{10}$$

† Codes of all other rules can be obtained from the authors through E-mail.

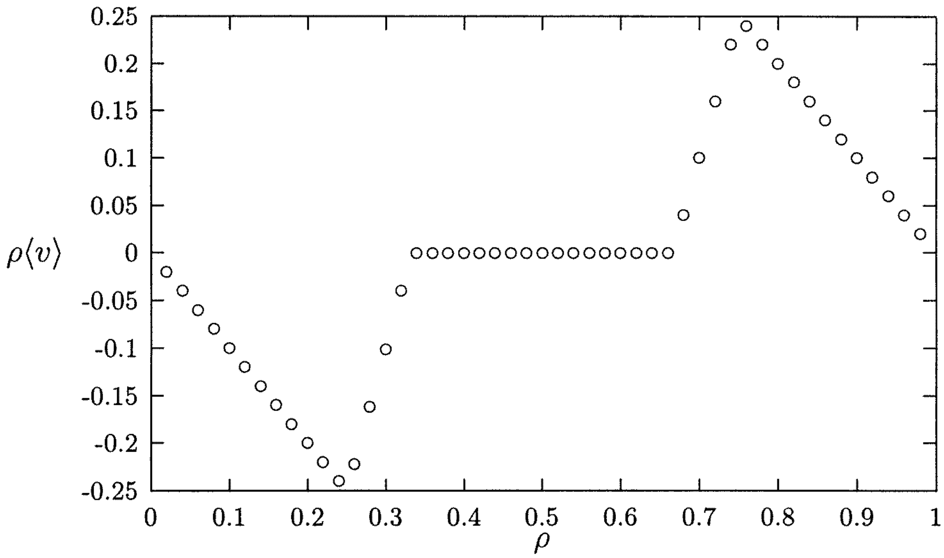

Figure 3. Fundamental diagram for rule 316 765 3058. Small circles represent numerical results.

is rather peculiar. As shown in figure 3, this rule emulates the identity only for $\rho \in [\frac{1}{3}, \frac{2}{3}]$. Rule 427 001 4080, obtained by reflection has identical properties. Note that the flow diagram (figure 3) is piecewise linear.

The four remaining rules are similar to the four-input rules 60 200 and 48 770 in the sense that they have similar flow diagrams and that, for $\rho = \frac{1}{2}$, they mimic pseudorandom walkers. These rules are as follows.

- Rule 322 112 7170 ($r_l = 2$, $r_r = 2$), whose motion representation is

$$00\overset{\curvearrowleft}{0}1,\quad 1\overset{\circlearrowright}{1}11.\ 10\overset{\curvearrowleft}{0}1,\quad 10\overset{\circlearrowright}{1},\quad 1\overset{\curvearrowright}{1}0,\quad 1\overset{\curvearrowright}{1}10$$

and rule 393 708 6120 obtained by reflection.

- Rule 370 703 1748 ($r_l = 2$, $r_r = 2$), whose motion representation is

$$0\overset{\curvearrowleft}{0}10,\quad 00\overset{\circlearrowright}{1}1,\quad 1\overset{\circlearrowright}{1}00,\quad 01\overset{\circlearrowright}{1}1,\quad 10\overset{\circlearrowright}{1},\quad 1\overset{\curvearrowright}{1}01,\quad 11\overset{\circlearrowright}{1}1$$

and rule 416 291 200 obtained by reflection.

The fundamental diagrams of rules 322 112 7170 and 370 703 1748 are represented, respectively, in figures 4 and 5. Here again we verified that the corresponding stochastic processes are not Gaussian. We have found that their Hurst exponents are equal for all of them to 0.57 ± 0.02. We have no explanation why rules 322 112 7170 and 370 703 1748 should have the same exponent.

5. Approximate methods

The mean-field approximation, which neglects correlations in space and time, yields, for these systems, an exact but trivial result. Let $\rho(t)$ denotes the particles density at time t. To find the expression of $\rho(t+1)$ as a function of $\rho(t)$, we have to find all the preimages of 1 by the n-input rule f. According to (5) all these preimages contain at least one 1. Moreover, among all of the preimages containing exactly $k+1$ times the digit 1 ($0 \leqslant k \leqslant n-1$),

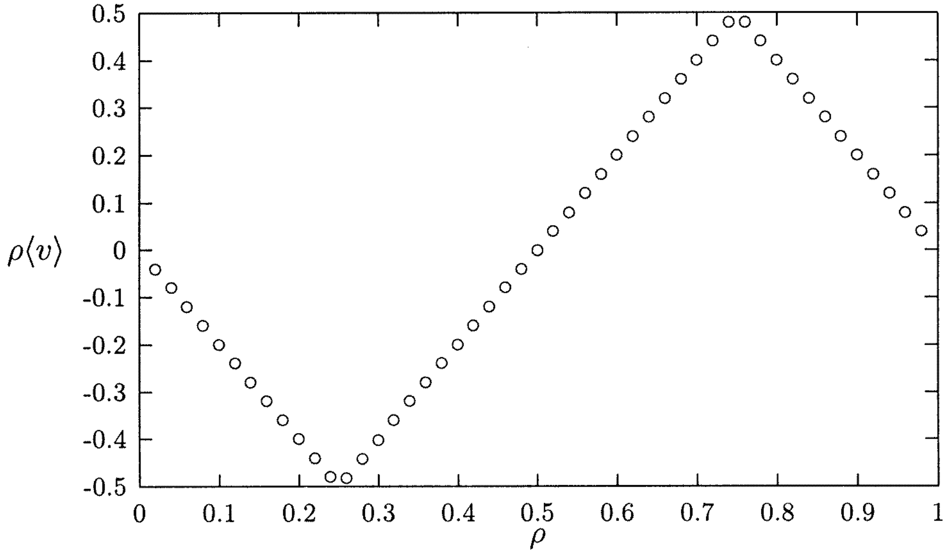

Figure 4. Fundamental diagram for rule 322 112 7170. Small circles represent numerical results.

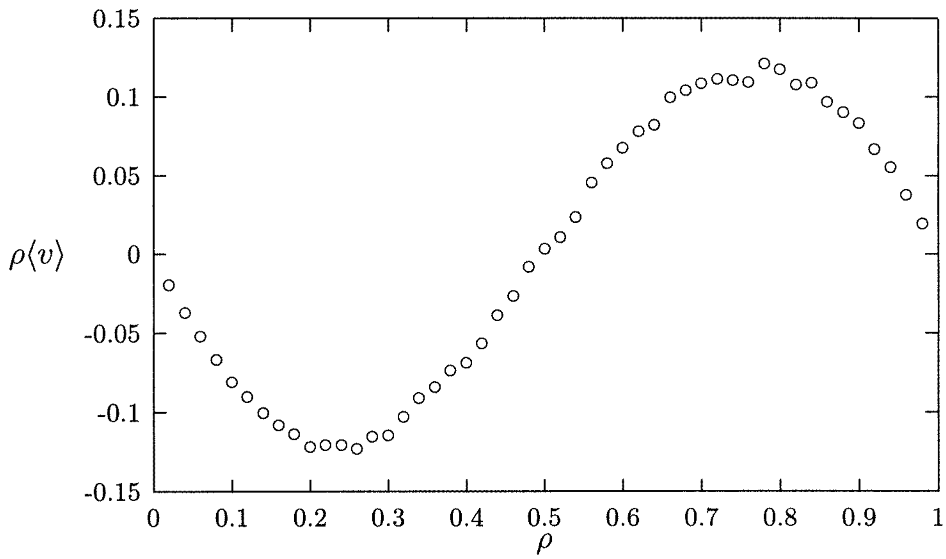

Figure 5. Fundamental diagram for rule 370 703 1748. Small circles represent numerical results.

according to the conditions (10) for $L = n$, only $\binom{n-1}{k}$ have a preimage equal to 1. Therefore,

$$\begin{aligned}\rho(t+1) &= \rho(t)\left(\sum_{k=0}^{n-1}\binom{n-1}{k}(\rho(t))^k(1-\rho(t))^{n-k-1}\right)\\ &= \rho(t)(\rho(t)+(1-\rho(t)))^{n-1}\\ &= \rho(t)\end{aligned}$$

which expresses that density is conserved.

There exists a variety of other approximate methods which, taking into account short-range correlations, improve the prediction of the mean-field approximation. Instead of expressing the evolution of the CA in terms of one-block probabilities, they express it in terms of n-block probabilities (Gutowitz *et al* 1987). For example, in the case of four-input rules, the evolution of the two-block probability distribution is given by

$$P(a_1a_2) = \sum_{b_0,b_1,b_2,b_3,b_4 \in \{0,1\}} w(a_1a_2|b_0b_1b_2b_3b_4)P(b_0b_1b_2b_3b_4)$$

where $P(a_1a_2)$ is the probability of block a_1a_2, and

$$w(a_1a_2|b_0b_1b_2b_3b_4) = w(a_1|b_0b_1b_2b_3)w(a_2|b_1b_2b_3b_4)$$

is the conditional probability that the four-input rule maps the five-block $b_0b_1b_2b_3b_4$ into the two-block a_1a_2. This equation is exact. The approximation consists of replacing the five-block probability $P(b_0b_1b_2b_3b_4)$ in terms of two-block probabilities. That is,

$$P(b_0b_1b_2b_3b_4) = \frac{P(b_0b_1)P(b_1b_2)P(b_2b_3)P(b_3b_4)}{(P(b_10) + P(b_11))(P(b_20) + P(b_21))(P(b_30) + P(b_31))}.$$

We applied this method up to approximation of order 3 (mean field being order 1) to four-input rule 60 200. The results are not exact, but for the flow diagram the agreement with our numerical results is extremely good (figure 2).

6. Conclusion

We have established necessary and sufficient conditions to be satisfied by any one-dimensional CA rule conserving the number of active sites. This result has been used to determine all of the four- and five-input one-dimensional CA rules possessing this property. These rules express the evolution of one-dimensional systems of interacting particles whose number is conserved. Simple deterministic highway traffic rules belong to that class of rules. These rules are a natural generalization of deterministic traffic rules already studied. We have studied in more detail (flow diagram, local structure approximation) some of our rules allowing motion of the particles in both directions. When the particle density is equal to $\frac{1}{2}$, these rules mimic the evolution of pseudorandom walkers. Numerical evidence seems to indicate that the motion of these walkers is non-Gaussian.

References

Boccara N, Goles E, Martínez S and Picco P 1993 *Cellular Automata and Cooperative Phenomena (Proc. Workshop, Les Houches)* (Dordrecht: Kluwer Academic)

Denning P, Dennis J and Qualitz J 1978 *Machines, Languages and Computation* (Englewood Cliffs, NJ: Prentice-Hall)

Farmer D, Toffoli T and Wolfram S 1984 *Cellular Automata (Proc. Interdisciplinary Workshop, Los Alamos)* (Amsterdam: North-Holland)

Feder J 1988 *Fractals* (New York: Plenum)

Fukś H and Boccara N 1998 *Int. J. Mod. Phys.* C **9** 1

Fukui M and Ishibashi Y 1995 *J. Phys. Soc. Japan* **64** 1868

Gutowitz H 1990 *Cellular Automata: Theory and Experiments (Proc. Workshop Los Alamos)* (Amsterdam: North-Holland)

Gutowitz H A, Victor J D and Knight B W 1987 *Physica* **28D** 18

Hopcroft J E and Ullmam J A 1986 *Formal Languages and their Relation to Automata* (Reading, MA: Addison-Wesley)

Hurst H E 1951 *Trans. Am. Soc. Civil Eng.* **116** 770

Hurst H E, Black R P and Simaika Y N 1965 *Long-term Storage: An Experimental Study* (London: Constable)

Manneville P, Boccara N, Vichniac G and Bidaux R 1989 *Cellular Automata and Modeling of Complex Systems (Proc. Workshop Les Houches)* (Heidelberg: Springer)
Nagel K and Schreckenberg M 1992 *J. Physique* I **2** 2221
Weiss G H 1994 *Aspects and Applications of the Random Walk* (Amsterdam: North-Holland)
Wolfram S 1983 *Rev. Mod. Phys.* **55** 601
——1984 *Commun. Math. Phys.* **96** 15
——1994 *Cellular Automata and Complexity: Collected Papers* (Reading, MA: Addison-Wesley)